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Real Hypersurfaces in Complex Hyperbolic Two-Plane Grassmannians with Commuting Structure Jacobi Operators

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Abstract. In this paper, we introduce a new commuting condition between the structure Jacobi operator and symmetric (1,1)-type tensor field T, that is, $R_\xi \phi T = T R_\xi \phi$, where $T = A$ or $T = S$ for Hopf hypersurfaces in complex hyperbolic two-plane Grassmannians. Using simultaneous diagonalization for commuting symmetric operators, we give a complete classification of real hypersurfaces in complex hyperbolic two-plane Grassmannians with commuting condition, respectively.

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1. Introduction

It is one of the main topics in submanifold geometry to investigate immersed real hypersurfaces of homogeneous type in Hermitian symmetric spaces of rank 2 (HSS2) with certain geometric conditions. Understanding and classifying real hypersurfaces in HSS2 is one of important problems in differential geometry. One of these spaces is the complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ defined by the set of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Another one is the complex hyperbolic two-plane Grassmannian $G_2^*(\mathbb{C}^{m+2}) = SU_{2,m}/S(U_2{\cdot}U_m)$ defined by the set of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$.

These are typical examples of HSS2. Characterizing typical model spaces of real hypersurfaces under certain geometric conditions is one of our main

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interests in the classification theory in $G_2(\mathbb{C}^{m+2})$ or $SU_{2,m}/S(U_2{\cdot}U_m)$ (see [13,14]).

Our recent interest is the study by applying geometric conditions used in submanifolds in $G_2(\mathbb{C}^{m+2})$ to submanifolds in $SU_{2,m}/S(U_2{\cdot}U_m)$.

$G_2(\mathbb{C}^{m+2}) = SU_{2+m}/S(U_2{\cdot}U_m)$ has compact transitive group SU_{2+m}; however, $SU_{2,m}/S(U_2{\cdot}U_m)$ has noncompact indefinite transitive group $SU_{2,m}$. This distinction gives various remarkable results.

The complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$ is the unique noncompact, irreducible, Kähler and quaternionic Kähler manifold which is not a hyperkähler manifold.

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$. Let N be a local unit normal vector field on M. Since the complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ has the Kähler structure J, we may define a *Reeb vector field* $\xi = -JN$ and a 1-dimensional distribution $\mathcal{C}^{\perp} = \mathrm{Span}\{xi\}$.

Let $\mathcal{C}$ be the orthogonal complement of distribution $\mathcal{C}^{\perp}$ in T_pM at $p \in M$. It is the complex maximal subbundle of T_pM. Thus, the tangent space of M consists of the direct sum of $\mathcal{C}$ and $\mathcal{C}^{\perp}$ as follows: $T_pM = \mathcal{C} \oplus \mathcal{C}^{\perp}$. The real hypersurface M is said to be *Hopf* if $A\mathcal{C} \subset \mathcal{C}$, or equivalently, the Reeb vector field ξ is principal with principal curvature $\alpha = g(A\xi, \xi)$, where g denotes the metric. In this case, the principal curvature α is said to be a *Reeb curvature* of M.

From the quaternionic Kähler structure $\mathfrak{J} = \mathrm{Span}\{J_1, J_2, J_3\}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$, there naturally exist *almost contact 3-structure* vector fields $\xi_\nu = -J_\nu N$, $\nu = 1, 2, 3$. Let $\mathcal{Q}^{\perp} = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$. It is a 3-dimensional distribution in the tangent space T_pM of M at $p \in M$. In addition, $\mathcal{Q}$ stands for the orthogonal complement of $\mathcal{Q}^{\perp}$ in T_pM. It is the quaternionic maximal subbundle of T_pM. Thus, the tangent space of M can be split into $\mathcal{Q}$ and $\mathcal{Q}^{\perp}$ as follows: $T_pM = \mathcal{Q} \oplus \mathcal{Q}^{\perp}$.

Thus, we have considered two natural geometric conditions for real hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ such that the subbundles $\mathcal{C}$ and $\mathcal{Q}$ of TM are both invariant under the shape operator. Using these geometric conditions, we will use the results in Suh [13, Theorem 1].

On the other hand, a Jacobi field along geodesics of a given Riemannian manifold $(\bar{M}, \bar{g})$ plays an important role in the study of differential geometry. It satisfies a well-known differential equation which inspires Jacobi operators. The Jacobi operator with respect to a vector field X on $\bar{M}$ is defined by $(\bar{R}_X(Y))(p) = (\bar{R}(Y, X)X)(p)$, where $\bar{R}$ denotes the curvature tensor of $\bar{M}$ and X, Y denote any vector fields on $\bar{M}$. It is known to be a self-adjoint endomorphism on the tangent space $T_p\bar{M}$, $p \in \bar{M}$. Clearly, each tangent vector field X to $\bar{M}$ provides a Jacobi operator with respect to X. Thus, the Jacobi operator on a real hypersurface M of $\bar{M}$ with respect to ξ is said to be a *structure Jacobi operator* and will be denoted by R_ξ. The Riemannian curvature tensor of M (resp., $\bar{M}$) is denoted by R (resp., $\bar{R}$).

For a commuting problem concerned with the structure Jacobi operator R_ξ and the structure tensor ϕ of a Hopf hypersurface M in $G_2(\mathbb{C}^{m+2})$, that is, $R_\xi \phi A = AR_\xi \phi$, Lee, Suh and Woo [3] proved that a Hopf hypersurface M satisfying $R_\xi \phi A = AR_\xi \phi$ and $\xi\alpha = 0$ is locally congruent to an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$. Motivated by this result, we consider the same condition in the different ambient space, that is,

$$R_\xi \phi AX = AR_\xi \phi X \tag{C-1}$$

for any tangent vector field X on M in $SU_{2,m}/S(U_2{\cdot}U_m)$. The geometric meaning of $R_\xi \phi AX = AR_\xi \phi X$ can be explained in such a way that any eigenspace of R_ξ on the distribution $\mathcal{C} = \{X \in T_pM | X \perp \xi\}$, $p \in M$, is invariant under the shape operator A of M in $SU_{2,m}/S(U_2{\cdot}U_m)$. Then using [13, Theorem 1], we give a complete classification of Hopf hypersurfaces in $SU_{2,m}/S(U_2{\cdot}U_m)$ with $R_\xi \phi AX = AR_\xi \phi X$ as follows:

Theorem 1. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, satisfying $R_\xi \phi A = AR_\xi \phi$. If the Reeb curvature $\alpha = g(A\xi, \xi)$ is constant along the Reeb direction of the structure vector field ξ, then M is locally congruent to one of the following:*

(i) *A tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or*

(ii) *A horosphere in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.*

From the Riemannian curvature tensor R of M in $SU_{2,m}/S(U_2{\cdot}U_m)$, we can define the Ricci tensor S of M in such a way that

$$g(SX, Y) = \sum_{i=1}^{4m-1} g(R(e_i, X)Y, e_i),$$

where $\{e_1, \cdots, e_{4m-1}\}$ denotes a basis of the tangent space T_pM of M, $p{\in}M$, in $SU_{2,m}/S(U_2{\cdot}U_m)$ (see [15]). Then, we can consider another new commuting condition

$$R_\xi \phi SX = SR_\xi \phi X \tag{C-2}$$

for any tangent vector field X on M. That is, the operator $R_\xi \phi$ commutes with the Ricci tensor S.

Then by [13, Theorem 1], we also give another classification related to the Ricci tensor S of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ as follows:

Theorem 2. *Let M be a Hopf hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, satisfying $R_\xi \phi S = SR_\xi \phi$. If the smooth function $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then M is locally congruent to one of the following:*

(i) *A tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or*

(ii) *A horosphere in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.*

In this paper, we refer [10, 13–15] for Riemannian geometric structures of complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$.

2. The Complex Hyperbolic Two-Plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section, we summarize basic materials about complex hyperbolic two-plane Grassmann manifolds $SU_{2,m}/S(U_2{\cdot}U_m)$, for details we refer to [9,11,13,15]. The Riemannian symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$, which consists of all complex two-dimensional linear subspaces in indefinite complex Euclidean space $\mathbb{C}_2^{m+2}$, is a connected, simply connected, irreducible Riemannian symmetric space of noncompact type and with rank two. Let $G = SU_{2,m}$ and $K = S(U_2{\cdot}U_m)$, and denote by $\mathfrak{g}$ and $\mathfrak{k}$ the corresponding Lie algebra of the Lie group G and K, respectively. Let B be the Killing form of $\mathfrak{g}$ and denote by $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. The Cartan involution $\theta \in \mathrm{A}ut(\mathfrak{g})$ on $\mathfrak{su}_{2,m}$ is given by $\theta(A) = I_{2,m}AI_{2,m}$, where

$$I_{2,m} = \begin{pmatrix} -I_2 & 0_{2,m} \\ 0_{m,2} & I_m \end{pmatrix},$$

I_2 and I_m denote the identity 2×2-matrix and $m \times m$-matrix, respectively. Then, $< X, Y >= -B(X, \theta Y)$ becomes a positive definite $\mathrm{A}d(K)$-invariant inner product on $\mathfrak{g}$. Its restriction to $\mathfrak{p}$ induces a metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$, which is also known as the Killing metric on $SU_{2,m}/S(U_2{\cdot}U_m)$. Throughout this paper, we consider $SU_{2,m}/S(U_2{\cdot}U_m)$ together with this particular Riemannian metric g.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{su}_2 \oplus \mathfrak{su}_m \oplus \mathfrak{u}_1$, where $\mathfrak{u}_1$ is the one-dimensional center of $\mathfrak{k}$. The adjoint action of $\mathfrak{su}_2$ on $\mathfrak{p}$ induces the quaternionic Kähler structure $\mathfrak{J}$ on $SU_{2,m}/S(U_2{\cdot}U_m)$, and the adjoint action of

$$Z = \begin{pmatrix} \frac{mi}{m+2} I_2 & 0_{2,m} \\ 0_{m,2} & \frac{-2i}{m+2} I_m \end{pmatrix} \in \mathfrak{u}_1$$

induces the Kähler structure J on $SU_{2,m}/S(U_2{\cdot}U_m)$. By construction, J commutes with each almost Hermitian structure J_ν in $\mathfrak{J}$ for $\nu = 1, 2, 3$. Recall that a canonical local basis $\{J_1, J_2, J_3\}$ of a quaternionic Kähler structure $\mathfrak{J}$ consists of three almost Hermitian structures J_1, J_2, J_3 in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is to be taken modulo 3. The tensor field JJ_ν, which is locally defined on $SU_{2,m}/S(U_2{\cdot}U_m)$, is self-adjoint and satisfies $(JJ_\nu)^2 = I$ and $tr(JJ_\nu) = 0$, where I is the identity transformation. For a nonzero tangent vector X, we define $\mathbb{R}X = \{\lambda X | \lambda \in \mathbb{R}\}$, $\mathbb{C}X = \mathbb{R}X \oplus \mathbb{R}JX$, and $\mathbb{H}X = \mathbb{R}X \oplus \mathfrak{J}X$.

We identify the tangent space $T_oSU_{2,m}/S(U_2{\cdot}U_m)$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ at o with $\mathfrak{p}$ in the usual way. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Since $SU_{2,m}/S(U_2{\cdot}U_m)$ has rank two, the dimension of any such subspace is two. Every nonzero tangent vector $X \in T_oSU_{2,m}/S(U_2{\cdot}U_m) \cong \mathfrak{p}$ is contained in some maximal abelian subspace of $\mathfrak{p}$. Generically this subspace is uniquely determined by X, in which case X is called regular. If there exist more than one maximal abelian subspaces of $\mathfrak{p}$ containing X, then X is called singular. There is a simple and useful characterization of the singular tangent vectors:

A nonzero tangent vector $X \in \mathfrak{p}$ is singular if and only if $JX \in \mathfrak{J}X$ or $JX \perp \mathfrak{J}X$.

Up to scaling there exists a unique $SU_{2,m}$-invariant Riemannian metric g on $SU_{2,m}/S(U_2{\cdot}U_m)$. Equipped with this metric, $SU_{2,m}/S(U_2{\cdot}U_m)$ is a Riemannian symmetric space of rank two which is both Kähler and quaternionic Kähler. For computational reasons, we normalize g such that the minimal sectional curvature of $(SU_{2,m}/S(U_2{\cdot}U_m), g)$ is -4. The sectional curvature K of the noncompact symmetric space $SU_{2,m}/S(U_2{\cdot}U_m)$ equipped with the Killing metric g is bounded by $-4{\le}K{\le}0$. The sectional curvature -4 is obtained for all two-planes $\mathbb{C}X$ when X is a non-zero vector with $JX \in \mathfrak{J}X$.

When $m = 1$, $G_2^*(\mathbb{C}^3) = SU_{1,2}/S(U_1{\cdot}U_2)$ is isometric to the two-dimensional complex hyperbolic space $\mathbb{C}H^2$ with constant holomorphic sectional curvature -4.

When $m = 2$, we note that the isomorphism $SO(4, 2) \simeq SU_{2,2}$ yields an isometry between $G_2^*(\mathbb{C}^4) = SU_{2,2}/S(U_2{\cdot}U_2)$ and the indefinite real Grassmann manifold $G_2^*(\mathbb{R}_2^6)$ of oriented two-dimensional linear subspaces of an indefinite Euclidean space $\mathbb{R}_2^6$. For this reason we assume $m \geq 3$ from now on, although many of the subsequent results also hold for $m = 1, 2$.

From now on, hereafter X,Y and Z always stand for any tangent vector fields on M.

The Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ is locally given by

$$\begin{aligned}
-2\bar{R}(X,Y)Z &= g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\
&\quad - g(JX,Z)JY - 2g(JX,Y)JZ \\
&\quad + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\
&\quad + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},
\end{aligned}$$

where $\{J_1, J_2, J_3\}$ is any canonical local basis of $\mathfrak{J}$.

3. Fundamental formulas in $SU_{2,m}/S(U_2{\cdot}U_m)$

In this section, we derive some basic formulas and the Codazzi equation for a real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ (see [13–15]).

Let M be a real hypersurface in complex hyperbolic two-plane Grassmannian $SU_{2,m}/S(U_2{\cdot}U_m)$, that is, a submanifold in $SU_{2,m}/S(U_2{\cdot}U_m)$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Levi Civita covariant derivative of (M, g). We denote by $\mathcal{C}$ and $\mathcal{Q}$ the maximal complex and quaternionic subbundle of the tangent bundle TM of M, respectively. Now let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{3.1}$$

for any tangent vector field X of a real hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$, where ϕX denotes the tangential component of JX and N a unit normal vector field of M in $SU_{2,m}/S(U_2{\cdot}U_m)$.

From the Kähler structure J of $SU_{2,m}/S(U_2{\cdot}U_m)$ there exists an almost contact metric structure (ϕ,ξ,η,g) induced on M in such a way that

$$\phi^2X=-X+\eta(X)\xi,\quad \eta(\xi)=1,\quad \phi\xi=0,\quad \eta(X)=g(X,\xi) \tag{3.2}$$

for any vector field X on M. Furthermore, let $\{J_1,J_2,J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then, the quaternionic Kähler structure J_ν of $SU_{2,m}/S(U_2{\cdot}U_m)$, together with the condition $J_\nu J_{\nu+1}=J_{\nu+2}=-J_{\nu+1}J_\nu$ in section 1, induces an almost contact metric 3-structure $(\phi_\nu,\xi_\nu,\eta_\nu,g)$ on M as follows:

$$\begin{aligned}
\phi_\nu^2X&=-X+\eta_\nu(X)\xi_\nu,\quad \eta_\nu(\xi_\nu)=1,\quad \phi_\nu\xi_\nu=0,\\
\phi_{\nu+1}\xi_\nu&=-\xi_{\nu+2},\quad \phi_\nu\xi_{\nu+1}=\xi_{\nu+2},\\
\phi_\nu\phi_{\nu+1}X&=\phi_{\nu+2}X+\eta_{\nu+1}(X)\xi_\nu,\\
\phi_{\nu+1}\phi_\nu X&=-\phi_{\nu+2}X+\eta_\nu(X)\xi_{\nu+1}
\end{aligned} \tag{3.3}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J=JJ_\nu$, $\nu=1,2,3$ in Sect. 2 and (3.1), the relation between these two contact metric structures (ϕ,ξ,η,g) and $(\phi_\nu,\xi_\nu,\eta_\nu,g)$, $\nu=1,2,3$, can be given by

$$\begin{aligned}
\phi\phi_\nu X&=\phi_\nu\phi X+\eta_\nu(X)\xi-\eta(X)\xi_\nu,\\
\eta_\nu(\phi X)&=\eta(\phi_\nu X),\quad \phi\xi_\nu=\phi_\nu\xi.
\end{aligned} \tag{3.4}$$

On the other hand, from the parallelism of Kähler structure J, that is, $\widetilde{\nabla}J=0$ and the quaternionic Kähler structure $\mathfrak{J}$, together with Gauss and Weingarten formulas, it follows that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX,Y)\xi,\quad \nabla_X\xi=\phi AX, \tag{3.5}$$

$$\nabla_X\xi_\nu=q_{\nu+2}(X)\xi_{\nu+1}-q_{\nu+1}(X)\xi_{\nu+2}+\phi_\nu AX, \tag{3.6}$$

$$\begin{aligned}
(\nabla_X\phi_\nu)Y\ &=-q_{\nu+1}(X)\phi_{\nu+2}Y+q_{\nu+2}(X)\phi_{\nu+1}Y+\eta_\nu(Y)AX\\
&\quad -g(AX,Y)\xi_\nu,
\end{aligned} \tag{3.7}$$

for some 1-forms q_1, q_2, q_3 on M.

Combining these formulas, we find the following:

$$\begin{aligned}
\nabla_X(\phi_\nu\xi)=\nabla_X(\phi\xi_\nu)&=(\nabla_X\phi)\xi_\nu+\phi(\nabla_X\xi_\nu)\\
&=q_{\nu+2}(X)\phi_{\nu+1}\xi-q_{\nu+1}(X)\phi_{\nu+2}\xi+\phi_\nu\phi AX\\
&\quad -g(AX,\xi)\xi_\nu+\eta(\xi_\nu)AX.
\end{aligned} \tag{3.8}$$

Finally, using the explicit expression for the Riemannian curvature tensor $\bar{R}$ of $SU_{2,m}/S(U_2{\cdot}U_m)$ in [14], the Codazzi equation takes the form

$$\begin{aligned}-2(\nabla_X A)Y + 2(\nabla_Y A)X &= \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\\ &\quad + \sum_{\nu=1}^{3}\left\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\right\}\\ &\quad + \sum_{\nu=1}^{3}\left\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu\phi X\right\}\\ &\quad + \sum_{\nu=1}^{3}\left\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\right\}\xi_\nu, \end{aligned} \tag{3.9}$$

for any vector fields X and Y on M.

On the other hand, by differentiating $A\xi = \alpha\xi$ and using (3.9), we get the following

$$\begin{aligned} & g(\phi X, Y) - \sum_{\nu=1}^{3}\{\eta_\nu(X)\eta_\nu(\phi Y) - \eta_\nu(Y)\eta_\nu(\phi X) - g(\phi_\nu X, Y)\eta_\nu(\xi)\}\\ &= g((\nabla_X A)Y - (\nabla_Y A)X, \xi)\\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X)\\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned} \tag{3.10}$$

Putting $X = \xi$ gives

$$Y\alpha = (\xi\alpha)\eta(Y) + 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y). \tag{3.11}$$

Then, substituting (3.11) into (3.10), we have

$$\begin{aligned} A\phi AY &= \frac{\alpha}{2}(A\phi + \phi A)Y + \sum_{\nu=1}^{3}\left\{\eta(Y)\eta_\nu(\xi)\phi\xi_\nu + \eta_\nu(\xi)\eta_\nu(\phi Y)\xi\right\}\\ &\quad -\frac{1}{2}\phi Y - \frac{1}{2}\sum_{\nu=1}^{3}\left\{\eta_\nu(Y)\phi\xi_\nu + \eta_\nu(\phi Y)\xi_\nu + \eta_\nu(\xi)\phi_\nu Y\right\}. \end{aligned} \tag{3.12}$$

By differentiating and using (3.4)–(3.6), we have

$$\begin{aligned} \nabla_X(\text{grad }\alpha) &= X(\xi\alpha)\xi + (\xi\alpha)\phi AX\\ &\quad - 2\sum_{\nu=1}^{3}\Big\{q_{\nu+2}(X)\eta_{\nu+1}(\xi) - q_{\nu+1}(X)\eta_{\nu+2}(\xi) + 2\eta_\nu(\phi AX)\Big\}\phi\xi_\nu\\ &\quad - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\Big\{-q_{\nu+1}(X)\phi_{\nu+2}\xi + q_{\nu+2}(X)\phi_{\nu+1}\xi + \eta_\nu(\xi)AX\\ &\qquad -g(AX,\xi)\xi_\nu + \phi_\nu\phi AX\Big\} \end{aligned}$$

$$
\begin{aligned}
&= X(\xi\alpha)\xi + (\xi\alpha)\phi AX - 4\sum_{\nu=1}^{3}\eta_\nu(\phi AX)\phi\xi_\nu \\
&\quad - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\Big\{\eta_\nu(\xi)AX - g(AX,\xi)\xi_\nu + \phi_\nu\phi AX\Big\}.
\end{aligned}
$$

By taking the skew-symmetric part to the above equation, we have

$$
\begin{aligned}
0 &= X(\xi\alpha)\eta(Y) - Y(\xi\alpha)\eta(X) + (\xi\alpha)g\big((A\phi+\phi A)X,Y\big) \\
&\quad -4\sum_{\nu=1}^{3}\Big\{\eta_\nu(\phi AX)g(\phi\xi_\nu,Y) - \eta_\nu(\phi AY)g(\phi\xi_\nu,X)\Big\} \\
&\quad +2\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\Big\{\eta(X)\eta_\nu(Y) - \eta(Y)\eta_\nu(X)\Big\} \\
&\quad -2\sum_{\nu=1}^{3}\eta_\nu(\xi)\Big\{g(\phi_\nu\phi AX,Y) - g(\phi_\nu\phi AY,X)\Big\}.
\end{aligned}
$$

From this, by putting $X=\xi$ we have the following

$$
Y(\xi\alpha) = \xi(\xi\alpha)\eta(Y) + 2\alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(Y) - 2\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(AY). \tag{3.13}
$$

From this, if we assume that $\xi\alpha=0$, then it follows that

$$
\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(AX) = \alpha\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(X).
$$

Lemma 1. *Let M be a Hopf real hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$. If the principal curvature α is constant along the direction of ξ, then the distribution $\mathcal{Q}$ or $\mathcal{Q}^\perp$ component of the structure vector field ξ is invariant by the shape operator.*

4. Proof of Theorem 1

Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with

$$
R_\xi\phi AX = AR_\xi\phi X. \tag{C-1}
$$

The structure Jacobi operator R_ξ of M is defined by $R_\xi X = R(X,\xi)\xi$ for any tangent vector $X\in T_pM$, $p\in M$ (see [1,7]).

Then for any tangent vector field X on M in $SU_{2,m}/S(U_2{\cdot}U_m)$, we calculate the structure Jacobi operator R_ξ

$$
\begin{aligned}
2R_\xi(X) &= -X + \eta(X)\xi + \sum_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu - \eta(X)\eta_\nu(\xi)\xi_\nu \\
&\quad +3\eta_\nu(\phi X)\phi_\nu\xi + \eta_\nu(\xi)\phi_\nu\phi X\} + 2\alpha AX - 2\eta(AX)A\xi,
\end{aligned} \tag{4.1}
$$

where α denotes the Reeb curvature defined by $g(A\xi,\xi)$.

Lemma 2. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with the commuting condition $R_\xi\phi AX = AR_\xi\phi X$. If the smooth function α is constant along the direction of ξ on M, then the Reeb vector field ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof. To prove this lemma, without loss of generality, ξ may be written as

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{*}$$

where X_0 (resp., ξ_1) is a unit vector in $\mathcal{Q}$ (resp., $\mathcal{Q}^\perp$) and $\eta(X_0)\eta(\xi_1) \neq 0$.

From (*) and $\phi\xi = 0$, we have

$$\begin{cases} \phi X_0 = -\eta(\xi_1)\phi_1 X_0, \\ \phi\xi_1 = \phi_1\xi = \eta(X_0)\phi_1 X_0, \\ \phi_1\phi X_0 = \eta_1(\xi)X_0. \end{cases} \tag{4.2}$$

Let $\mathfrak{U} = \{p \in M \,|\, \alpha(p) \neq 0\}$ be an open subset of M. From now on, we discuss our arguments on $\mathfrak{U}$.

By virtue of Lemma 1, $\xi\alpha = 0$ gives $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$. The equation (3.12) yields $\alpha A\phi X_0 = (\alpha^2 - 2\eta^2(X_0))\phi X_0$ by substituting $X = X_0$. Since α is non-vanishing on $\mathfrak{U}$, it becomes

$$A\phi X_0 = \sigma\phi X_0, \tag{4.3}$$

where $\sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}$.

From (4.2) and (4.3), we have

$$\begin{cases} R_\xi(X_0) = \alpha^2 X_0 - \alpha^2\eta(X_0)\xi, \\ R_\xi(\xi_1) = \alpha^2\xi_1 - \alpha^2\eta(\xi_1)\xi, \\ R_\xi(\phi X_0) = \big(\alpha^2 - 4\eta^2(X_0)\big)\phi X_0. \end{cases} \tag{4.4}$$

On $\mathfrak{U}$, substituting X by ϕX_0 into (C-1), we have

$$X_0 - \eta(X_0)\xi = 0, \tag{4.5}$$

which is a contradiction. Therefore, $\mathfrak{U} = \emptyset$, and thus it must be $p \in M - \mathfrak{U}$. Since the set $M - \mathfrak{U} = \mathrm{Int}(M - \mathfrak{U}) \cup \partial(M - \mathfrak{U})$, we consider the following two cases. Here Int (resp., ∂) denotes the interior (resp., the boundary) of $(M - \mathfrak{U})$.

- **Case 1** $p \in \mathrm{Int}(M - \mathfrak{U})$.

  If $p \in \mathrm{Int}(M - \mathfrak{U})$ i.e., $\alpha(p) = 0$, then it trivially holds by (3.11).

- **Case 2** $p \in \partial(M - \mathfrak{U})$.

  Since $p \in \partial(M - \mathfrak{U})$, there exists a sequence of points p_n such that $p_n \to p$ with $\alpha(p) = 0$ and $\alpha(p_n) \neq 0$. Such a sequence will have an infinite subsequence where $\eta(\xi_1) = 0$ (in which case $\xi \in \mathcal{Q}$ at p, by the continuity) or an infinite subsequence where $\eta(X_0) = 0$ (in which case $\xi \in \mathcal{Q}^\perp$ at p).

  Accordingly, we get a complete proof of our lemma. □

From Lemma 2, we consider the case that ξ belongs to the distribution $\mathcal{Q}^\perp$. Thus without loss of generality, we may put $\xi = \xi_1$.

Differentiating $\xi = \xi_1$ along any direction $X \in TM$ and using (3.5) and (3.6), it gives us

$$2\eta_3(AX)\xi_2 - 2\eta_2(AX)\xi_3 + \phi_1 AX - \phi AX = 0. \tag{4.6}$$

Then, using the symmetric (resp., skew-symmetric) property of the shape operator A (resp., the structure tensor field ϕ), we also obtain

$$2\eta_3(X)A\xi_2 - 2\eta_2(X)A\xi_3 + A\phi_1 X - A\phi X = 0. \tag{4.7}$$

Applying ϕ_1 to (4.6), it implies

$$2\eta_3(AX)\xi_3 + 2\eta_2(AX)\xi_2 - AX + \alpha\eta(X)\xi - \phi_1\phi AX = 0. \tag{4.8}$$

On the other hand, replacing $X = \phi X$ into (4.6), we have

$$-2\eta_2(X)A\xi_2 - 2\eta_3(X)A\xi_3 + A\phi_1\phi X - AX - \alpha\eta(X)\xi = 0. \tag{4.9}$$

Lemma 3. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, satisfying $R_\xi\phi A = AR_\xi\phi$. If the Reeb vector field ξ belongs to the distribution $\mathcal{Q}^\perp$, then the shape operator A commutes with the structure tensor field ϕ.*

Proof. Applying $\xi = \xi_1$ into right-hand side (resp., left-hand side) of (C-1), we get

$$\begin{aligned} 2R_\xi\phi AX &= -A\phi X + 2\alpha A^2\phi X - 2\eta_3(X)A\xi_2 + 2\eta_2(X)A\xi_3 - A\phi_1 X,\\ 2AR_\xi\phi X &= -\phi AX + 2\alpha A\phi AX - 2\eta_3(AX)\xi_2 + 2\eta_2(AX)\xi_3 - \phi_1 AX. \end{aligned}$$

Combining (4.6) and (4.7), the above equations become

$$\begin{aligned} R_\xi\phi AX &= -A\phi X + \alpha A^2\phi X,\\ AR_\xi\phi X &= -\phi AX + \alpha A\phi AX. \end{aligned}$$

Hence, (C-1) is equivalent to

$$A\phi - \phi A = \alpha A(A\phi - \phi A) \tag{4.10}$$

Taking the symmetric part of (4.10), we have

$$A\phi - \phi A = \alpha(A\phi - \phi A)A. \tag{4.11}$$

From this, the proof can be divided into the following three cases:

First, let us consider the open subset $\mathfrak{U} = \{p \in M \,|\, \alpha(p) \neq 0\}$ of M. Naturally we can apply (4.10) and (4.11) on the open subset $\mathfrak{U}$.

$$(A\phi - \phi A)AX = A(A\phi - \phi A)X.$$

Since the shape operator A and the tensor $A\phi - \phi A$ are both symmetric operators and commute with each other, there exists a common orthonormal basis $\{E_i\}_{i=1,\ldots,4m-1}$ which gives a simultaneous diagonalization. Specifically, we have

$$\begin{aligned} AE_i &= \lambda_i E_i, & (4.12)\\ (A\phi - \phi A)E_i &= \beta_i E_i. & (4.13) \end{aligned}$$

Taking the inner product with E_i into (4.13), we have

$$\beta_i g(E_i, E_i) = g\big((A\phi - \phi A)E_i, E_i\big) = -2\lambda_i g(\phi E_i, E_i) = 0. \tag{4.14}$$

Since $g(E_i, E_i) = 1$, $\beta_i = 0$ for all $i = 1, 2, ..., 4m-1$. Hence $A\phi X = \phi AX$ for any tangent vector field X on $\mathfrak{U}$.

Next, if $p \in \mathrm{Int}(M - \mathfrak{U})$, then $\alpha(p) = 0$. From this, the equation (4.11) gives $(A\phi - \phi A)X(p) = 0$.

Finally, let us assume that $p \in \partial(M - \mathfrak{U})$, where $\partial(M - \mathfrak{U})$ is the boundary of $M - \mathfrak{U}$. Then there exists a subsequence $\{p_n\} \subset \mathfrak{U}$ such that $p_n \to p$. Since $(A\phi - \phi A)X(p_n) = 0$ on the open subset $\mathfrak{U}$ in M, by the continuity we also get $(A\phi - \phi A)X(p) = 0$.

Summing up these observations, it is natural that the shape operator A commutes with the structure tensor field ϕ under our assumption. $\square$

By [11], we assert M with the assumptions given in Lemma 3 is locally congruent to one of the following hypersurfaces:

($\mathcal{T}_A$) a tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or,

($\mathcal{H}_A$) a horosphere in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.

In [11], Suh gave some information related to the shape operator A of $\mathcal{T}_A$ and $\mathcal{H}_A$ as follows:

Proposition A. *Let M be a connected real hypersurface in complex hyperbolic two-plane Grassamannian $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \in \mathfrak{J}N$, then one of the following statements holds:*

($\mathcal{T}_A$) *M has exactly four distinct constant principal curvatures*

$$\alpha = 2\coth(2r), \quad \beta = \coth(r), \quad \lambda_1 = \tanh(r), \quad \lambda_2 = 0,$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = \mathcal{C} \ominus \mathcal{Q}, \quad T_{\lambda_1} = E_{-1}, \quad T_{\lambda_2} = E_{+1}.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are complex (with respect to J) and totally complex (with respect to $\mathfrak{J}$).

($\mathcal{H}_A$) *M has exactly three distinct constant principal curvatures*

$$\alpha = 2, \quad \beta = 1, \quad \lambda = 0$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus \mathcal{C}, \quad T_\beta = (\mathcal{C} \ominus \mathcal{Q}) \oplus E_{-1}, \quad T_\lambda = E_{+1}.$$

Here, E_{+1} and E_{-1} are the eigenbundles of $\phi\phi_1|_{\mathcal{Q}}$ with respect to the eigenvalues $+1$ and -1, respectively.

Since the symmetric tensor $A\phi - \phi A$ vanishes identically on $\mathcal{T}_A$ (resp. $\mathcal{H}_A$), it trivially satisfies (4.10). Hence, we assert that $\mathcal{T}_A$ (resp., $\mathcal{H}_A$) in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ has the our commuting condition (C-1) (see [11]).

Next, due to Lemma 2, let us suppose that $\xi \in \mathcal{Q}$ (i.e., $JN \perp \mathfrak{J}N$).

By virtue of the result in [13], we assert that a Hopf hypersurface M in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying the hypotheses in Theorem 1 is locally congruent to one of the following real hypersurfaces

($\mathcal{T}_B$) An open part of a tube around a totally geodesic quaternionic hyperbolic space $\mathbb{H}H^n$ in $SU_{2,2n}/S(U_2U_{2n})$, $m = 2n$,

($\mathcal{H}_B$) An open part of a horosphere in $SU_{2,m}/S(U_2U_m)$ whose center at infinity is singular and of type $JN \perp \mathfrak{J}N$, or

($\mathcal{E}$) The normal bundle νM of M consists of singular tangent vectors of type $JX \perp \mathfrak{J}X$,

when $\xi \in \mathcal{Q}$. Hereafter, the model spaces of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ are denoted by M_B. Let us check whether the shape operator A of model spaces is M_B that satisfy our conditions, conversely. To do this, let us introduce the following proposition given by Suh [13].

Proposition B. *Let M be a connected hypersurface in $SU_{2,m}/S(U_2U_m)$, $m \geq 3$. Assume that the maximal complex subbundle $\mathcal{C}$ of TM and the maximal quaternionic subbundle $\mathcal{Q}$ of TM are both invariant under the shape operator of M. If $JN \perp \mathfrak{J}N$, then one of the following statements holds:*

($\mathcal{T}_B$) *M has five (four for $r = \sqrt{2}\tanh^{-1}(1/\sqrt{3})$ in which case $\alpha = \lambda_2$) distinct constant principal curvatures*

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r), \quad \beta = \sqrt{2}\coth(\sqrt{2}r), \quad \gamma = 0,$$
$$\lambda_1 = \frac{1}{\sqrt{2}}\tanh(\frac{1}{\sqrt{2}}r), \quad \lambda_2 = \frac{1}{\sqrt{2}}\coth(\frac{1}{\sqrt{2}}r),$$

and the corresponding principal curvature spaces are

$$T_\alpha = TM \ominus \mathcal{C},\ T_\beta = TM \ominus \mathcal{Q},\ T_\gamma = J(TM \ominus \mathcal{Q}) = JT_\beta.$$

The principal curvature spaces T_{λ_1} and T_{λ_2} are invariant under $\mathfrak{J}$ and are mapped onto each other by J. In particular, the quaternionic dimension of $SU_{2,m}/S(U_2U_m)$ must be even.

($\mathcal{H}_B$) *M has exactly three distinct constant principal curvatures*

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda = \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

($\mathcal{E}$) *M has at least four distinct principal curvatures, three of which are given by*

$$\alpha = \beta = \sqrt{2}, \quad \gamma = 0, \quad \lambda = \frac{1}{\sqrt{2}}$$

with corresponding principal curvature spaces

$$T_\alpha = TM \ominus (\mathcal{C} \cap \mathcal{Q}),\ T_\gamma = J(TM \ominus \mathcal{Q}),\ T_\lambda \subset \mathcal{C} \cap \mathcal{Q} \cap J\mathcal{Q}.$$

If μ is another (possibly nonconstant) principal curvature function, then $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$. Thus, the corresponding multiplicities are

$$m(\alpha) = 4, \quad m(\gamma) = 3, \quad m(\lambda), \quad m(\mu).$$

Let us assume that the structure Jacobi operator R_ξ of M_B satisfies the property (C-1).

The tangent space of M_B can be split into

$$TM = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5},$$

where $T_{\alpha_1} = [\xi]$, $T_{\alpha_2} = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$, $T_{\alpha_3} = \mathrm{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\}$ and $T_{\alpha_4} \oplus T_{\alpha_5}$ is the orthogonal complement of $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$ in TM. Since $\xi \in \mathcal{Q}$ and $\phi\phi_\nu\xi = \phi^2\xi_\nu = -\xi_\nu$, we have $R_\xi(\phi\xi_2) = -2\phi_2\xi$. From this and $\alpha_3 = 0$ for all M_B, our commuting condition (C-1) becomes

$$R_\xi\phi A\xi_2 - AR_\xi\phi\xi_2 = -2\alpha_2\phi\xi_2.$$

It implies that the eigenvalue α_2 vanishes, since $\phi\xi_2$ is a unit tangent vector field. But in Proposition B, for $\mathcal{T}_B$ (resp. $\mathcal{H}_B$ or $\mathcal{E}$) we see that the eigenvalue $\alpha_2 = \beta = \sqrt{2}\coth(\sqrt{2}r)$ (resp. $\alpha_2 = \alpha = \frac{1}{\sqrt{2}}$) is non-vanishing. This gives us a contradiction.

5. Proof of Theorem 2

In this section, using geometric quantities in [3–5, 13–15], we give a complete proof of Theorem 2. To prove it, we assume that M is a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$ with commuting structure Jacobi operator and Ricci tensor, that is,

$$(R_\xi\phi)SX = S(R_\xi\phi)X. \tag{C-2}$$

From the definition of the Ricci tensor and the fundamental formulas in [15, Section 2], the Ricci tensor S of M in $SU_{2,m}/S(U_2{\cdot}U_m)$ is given by

$$\begin{aligned} 2SX &= -(4m+7)X + 3\eta(X)\xi + 2hAX - 2A^2X \\ &\quad + \sum_{\nu=1}^{3}\{3\eta_\nu(X)\xi_\nu - \eta_\nu(\xi)\phi_\nu\phi X + \eta_\nu(\phi X)\phi_\nu\xi + \eta(X)\eta_\nu(\xi)\xi_\nu\}, \end{aligned} \tag{5.1}$$

where h denotes the trace of the shape operator A.

Using equations (C-2) and (5.1), we prove that the Reeb vector field ξ of M belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.

Lemma 4. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2{\cdot}U_m)$, $m \geq 3$, satisfying (C-2). If the principal curvature $\alpha = g(A\xi, \xi)$ is constant along the direction of ξ, then ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$.*

Proof. To prove this lemma, for some unit vectors $X_0 \in \mathcal{Q}$, $\xi_1 \in \mathcal{Q}^\perp$, we put

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1, \tag{*}$$

where $\eta(X_0)\eta(\xi_1) \neq 0$ is the assumption we will disprove in this proof by contradiction.

Let $\mathfrak{U} = \{p \in M \mid \alpha(p) \neq 0\}$ be the open subset of M. From now on, we discuss our arguments on $\mathfrak{U}$.

By virtue of Lemma 1, $\xi\alpha = 0$ gives $AX_0 = \alpha X_0$ and $A\xi_1 = \alpha\xi_1$. From (5.1), we have

$$\begin{cases} S\phi X_0 = \kappa\phi X_0, \\ SX_0 = (-2m - 4 + h\alpha - \alpha^2)X_0 + 2\eta(X_0)\xi, \\ S\xi_1 = (-2m - 2 + h\alpha - \alpha^2)\xi_1 + 2\eta_1(\xi)\xi, \\ S\xi = (-2m - 2 + h\alpha - \alpha^2)\xi + 2\eta_1(\xi)\xi_1, \end{cases} \tag{5.2}$$

where $\kappa := -2m - 4 + h\sigma - \sigma^2$ and $\sigma = \frac{\alpha^2 - 2\eta^2(X_0)}{\alpha}$ on $\mathfrak{U}$.

Put $X = \phi X_0$ into (C-2), we have

$$\kappa R_\xi(X_0) = SR_\xi(X_0). \tag{5.3}$$

Taking the inner product of (5.3) with ξ and using (4.4) and (5.2), we have $-2\alpha^2\eta^2(\xi_1)\eta(X_0) = 0$. It implies that $\mathfrak{U} = \emptyset$. Thus it must be $p \in M - \mathfrak{U}$. The set $M - \mathfrak{U} = \mathrm{Int}(M - \mathfrak{U}) \cup \partial(M - \mathfrak{U})$, where Int (resp., ∂) denotes the interior (resp., the boundary) of $M - \mathfrak{U}$, we consider the following two cases:

- **Case 1** $p \in \mathrm{Int}(M - \mathfrak{U})$

If $p \in \mathrm{Int}(M - \mathfrak{U})$, then $\alpha = 0$. Our lemma was proved on $\mathrm{Int}(M - \mathfrak{U})$ by the equation (3.11) and (*).

- **Case 2** $p \in \partial(M - \mathfrak{U})$

Since $p \in \partial(M - \mathfrak{U})$, there exists a sequence of points $p_n \in \mathfrak{U}$ such that $p_n \to p$ with $\alpha(p) = 0$ and $\alpha(p_n) \neq 0$. Such a sequence will have an infinite subsequence where $\eta(\xi_1) = 0$ (in which case $\xi \in \mathcal{Q}$ at p, by the continuity) or an infinite subsequence where $\eta(X_0) = 0$ (in which case $\xi \in \mathcal{Q}^\perp$ at p). Accordingly, we get a complete proof of the Lemma. $\square$

Now, we shall divide our consideration into two cases, ξ belongs to either the distribution $\mathcal{Q}$ or the distribution $\mathcal{Q}^\perp$, respectively.

Let us consider the case $\xi \in \mathcal{Q}^\perp$. We may put $\xi = \xi_1 \in \mathcal{Q}^\perp$ for the sake of convenience. Then, (5.1) is simplified:

$$\begin{aligned} 2SX &= -(4m+7)X + 7\eta(X)\xi + 2\eta_2(X)\xi_2 \\ &\quad + 2\eta_3(X)\xi_3 - \phi_1\phi X + 2hAX - 2A^2X. \end{aligned} \tag{5.4}$$

Replacing X by AX into (5.4) and using (4.8), we obtain

$$2SAX = -(4m+6)AX + 6\alpha\eta(X)\xi + 2hA^2X - 2A^3X \tag{5.5}$$

Applying the shape operator A to (5.4) and using (4.9), we get

$$2ASX = -(4m+6)AX + 6\alpha\eta(X)\xi + 2hA^2X - 2A^3X. \tag{5.6}$$

From (5.5) and (5.6), we see that the Ricci tensor S commutes with the shape operator A, that is,

$$SA = AS. \tag{5.7}$$

On the other hand, the equations (4.6) and (5.4) give us

$$\begin{aligned} &2\eta_3(SX)\xi_2 - 2\eta_2(SX)\xi_3 + \phi_1 SX - \phi SX \\ &\quad = (2m+4)\{2\eta_3(X)\xi_2 - 2\eta_2(X)\xi_3 + \phi X - \phi_1 X\} \\ &\quad := \mathrm{Rem}(X). \end{aligned} \tag{5.8}$$

Taking the symmetric part of (5.8), we obtain

$$2\eta_3(X)S\xi_2 - 2\eta_2(X)S\xi_3 + S\phi_1 X - S\phi X = \mathrm{Rem}(X). \tag{5.9}$$

Lemma 5. *Let M be a Hopf hypersurface in $SU_{2,m}/S(U_2 {\cdot} U_m)$ satisfying* (C-2)*. If $\xi \in \mathcal{Q}^\perp$, then $S\phi = \phi S$.*

Proof. By virtue of equation (5.8) and (5.9), we obtain the left and right sides of (C-2), respectively, as follows:

$$\begin{aligned} 2R_\xi\phi SX &= -\phi SX + 2\alpha A\phi SX - 2\eta_3(SX)\xi_2 + 2\eta_2(SX)\xi_3 - \phi_1 SX \\ &= -2\phi SX + 2\alpha A\phi SX - \mathrm{Rem}(X), \end{aligned}$$

and

$$\begin{aligned} 2SR_\xi\phi X &= -S\phi X + 2\alpha SA\phi X - 2\eta_3(X)S\xi_2 + 2\eta_2(X)S\xi_3 - S\phi_1 X \\ &= -2S\phi X + 2\alpha SA\phi X - \mathrm{Rem}(X). \end{aligned}$$

That is,

$$R_\xi\phi SX = -\phi SX + \alpha A\phi SX - \frac{1}{2}\mathrm{Rem}(X) \tag{5.10}$$

and

$$SR_\xi\phi X = -S\phi X + \alpha SA\phi X - \frac{1}{2}\mathrm{Rem}(X). \tag{5.11}$$

From these two equations, the condition (C-2) is equivalent to

$$\begin{aligned} (S\phi - \phi S)X &= \alpha(SA\phi - A\phi S)X \\ &= \alpha A(S\phi - \phi S)X, \end{aligned} \tag{5.12}$$

by virtue of our assertion that the shape operator A commutes with the Ricci tensor S.

Taking the symmetric part of (5.12), we have

$$(S\phi - \phi S)X = \alpha(S\phi - \phi S)AX \tag{5.13}$$

for all tangent vector fields X on M.

From (5.12) and (5.13), we know

$$\alpha A(S\phi - \phi S) = \alpha(S\phi - \phi S)A. \tag{5.14}$$

Let $\mathfrak{U} = \{p \in M \,|\, \alpha(p) \neq 0\}$ be an open subset of M. Then (5.14) implies the shape operator A and the symmetric tensor $S\phi - \phi S$ commute with each other on $\mathfrak{U}$. Hence, they are simultaneous diagonalizable and there exists a common orthonormal basis $\{E_1, E_2, ..., E_{4m-1}\}$ such that the shape operator A and the tensor $S\phi - \phi S$ both can be diagonalizable.

$$AE_i = \lambda_i E_i, \tag{5.15}$$

$$(S\phi - \phi S)E_i = \coprod \beta_i E_i, \tag{5.16}$$

Combining equations in (5.1), we get

$$S\phi X - \phi SX = hA\phi X - A^2\phi X - h\phi AX + \phi A^2 X. \tag{5.17}$$

Using (5.15), (5.16) and (5.17), we obtain

$$(S\phi - \phi S)E_i = hA\phi E_i - A^2\phi E_i - h\lambda_i\phi E_i + \lambda_i^2\phi E_i. \tag{5.18}$$

Taking the inner product with E_i into (5.18), we have

$$\beta_i g(E_i, E_i) = h\lambda_i g(\phi E_i, E_i) - \lambda_i^2 g(\phi E_i, E_i) - h\lambda_i g(\phi E_i, E_i) + \lambda_i^2 g(\phi E_i, E_i) = 0.$$

Since $g(E_i, E_i) = 1$, we get $\beta_i = 0$ for all $i = 1, 2, ..., 4m-1$. This is equivalent to $(S\phi - \phi S)E_i = 0$ for all $i = 1, 2, ..., 4m-1$. It follows that $S\phi X = \phi SX$ for any tangent vector field X on $\mathfrak{U}$. Next, if $p \in \mathrm{Int}(M - \mathfrak{U})$, then we see that $\alpha(p) = 0$. From this, the equation (5.12) gives $(S\phi - \phi S)$ vanishes identically on $\mathrm{Int}(M - \mathfrak{U})$.

Finally, let us assume that $p \in \partial(M - \mathfrak{U})$, where $\partial(M - \mathfrak{U})$ is the boundary of $M - \mathfrak{U}$. Then, there exists a subsequence $\{p_n\} \subset \mathfrak{U}$ such that $p_n \to p$. Since $(S\phi - \phi S)X(p_n) = 0$ on the open subset $\mathfrak{U}$ in M, by the continuity we also get $(S\phi - \phi S)X(p) = 0$. $\square$

By virtue of the result given by Suh in [14], we assert that *if $\xi \in \mathcal{Q}^\perp$, then a Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying* (C-2) *is locally congruent to one of the following hypersurfaces:*

($\mathcal{T}_A$) *a tube over a totally geodesic $SU_{2,m-1}/S(U_2{\cdot}U_{m-1})$ in $SU_{2,m}/S(U_2{\cdot}U_m)$ or,*

($\mathcal{H}_A$) *a horosphere in $SU_{2,m}/S(U_2{\cdot}U_m)$ whose center at infinity is singular and of type $JX \in \mathfrak{J}X$.*

Moreover, when $\xi \in \mathcal{Q}^\perp$, (C-2) is equivalent to (5.12). Since the symmetric tensor $(S\phi - \phi S)$ vanishes identically on $\mathcal{T}_A$ (resp. $\mathcal{H}_A$), it trivially satisfies (5.12). Hence, we assert that $\mathcal{T}_A$ (resp., $\mathcal{H}_A$) in complex hyperbolic two-plane Grassmannians $SU_{2,m}/S(U_2{\cdot}U_m)$ has our commuting condition (C-2) (see [14]).

When $\xi \in \mathcal{Q}$, a Hopf hypersurface M in $SU_{2,m}/S(U_2{\cdot}U_m)$ satisfying (C-2) is locally congruent to a hypersurface of type M_B by [13].

From now on, let us show whether model spaces M_B satisfy the condition (C-2) or not. Then, the tangent space of M_B can be split into

$$TM_B = T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3} \oplus T_{\alpha_4} \oplus T_{\alpha_5}.$$

where $T_{\alpha_1} = [\xi]$, $T_{\alpha_2} = \mathrm{span}\{\xi_1, \xi_2, \xi_3\}$, $T_{\alpha_3} = \mathrm{span}\{\phi\xi_1, \phi\xi_2, \phi\xi_3\}$ and $T_{\alpha_4} \oplus T_{\alpha_5}$ is the orthogonal complement of $T_{\alpha_1} \oplus T_{\alpha_2} \oplus T_{\alpha_3}$ in TM such that $JT_{\alpha_5} \subset T_{\alpha_4}$ (see [14]).

On T_pM_B, $p \in M_B$, the equations (5.1) and (4.1) are reduced to the following equations, respectively:

$$\begin{aligned} 2SX &= -(4m+7)X + 3\eta(X)\xi + 2hAX - 2A^2X \\ &\quad + \sum_{\nu=1}^{3}\{3\eta_\nu(X)\xi_\nu + \eta(\phi_\nu X)\phi_\nu\xi\}, \\ 2R_\xi(X) &= -X + \eta(X)\xi + 2\alpha AX - 2\alpha^2\eta(X)\xi \\ &\quad + \sum_{\nu=1}^{3}\{\eta_\nu(X)\xi_\nu + 3\eta_\nu(\phi X)\phi_\nu\xi\}. \end{aligned}$$

From [14, Proposition 5.1], we obtain the following

$$SX = \begin{cases} (-2m - 2 + h\alpha_1 - \alpha_1^2)\xi & \text{if } \ X = \xi \in T_{\alpha_1} \\ (-2m - 2 + h\alpha_2 - \alpha_2^2)\xi_\ell & \text{if } \ X = \xi_\ell \in T_{\alpha_2} \\ (-2m - 4)\phi\xi_\ell & \text{if } \ X = \phi\xi_\ell \in T_{\alpha_3} \\ (-2m - \frac{7}{2} + h\alpha_4 - \lambda_4^2)X & \text{if } \ X \in T_{\alpha_4} \\ (-2m - \frac{7}{2} + h\alpha_5 - \alpha_5^2)X & \text{if } \ X \in T_{\alpha_5} \end{cases} \tag{5.19}$$

$$R_\xi(X) = \begin{cases} 0 & \text{if } X = \xi \in T_{\alpha_1} \\ \alpha_1\alpha_2\xi_\ell & \text{if } X = \xi_\ell \in T_{\alpha_2} \\ (-2+\alpha_1\alpha_3)\phi\xi_\ell & \text{if } X = \phi\xi_\ell \in T_{\alpha_3} \\ (-\frac{1}{2}+\alpha_1\alpha_4)X & \text{if } X \in T_{\alpha_4} \\ (-\frac{1}{2}+\alpha_1\alpha_5)X & \text{if } X \in T_{\alpha_5}. \end{cases} \tag{5.20}$$

To check whether $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ model spaces satisfy (C-2) or not, we should verify the following equation vanishes for all cases.

$$G(X) := (R_\xi\phi)SX - S(R_\xi\phi)X. \tag{5.21}$$

Putting $X = \xi_1 \in T_{\alpha_3}$ into (5.21), we have $G(\xi_1) = -2(2+\alpha_2 h - {\alpha_2}^2)\phi\xi_1$ which derives

$$2 + \alpha_2 h - {\alpha_2}^2 = 0. \tag{5.22}$$

- **Case 1.** Tube $\mathcal{T}_B$

In this case, we get $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $\alpha_3 = \gamma = 0$, $\alpha_4 = \lambda$ and $\alpha_5 = \mu$. By calculation, we have $\lambda + \mu = \beta$ on $\mathcal{T}_B$. Thus we obtain $h = \alpha + 3\beta + (4n-4)(\lambda+\mu) = \alpha + (4n-1)\beta$, where $m = 2n$. Then (5.22) is equivalent to $4 + 2(2n-1)\beta^2 > 0$, which is a contradiction.

- **Case 2.** Horoshere $\mathcal{H}_B$

On $\mathcal{H}_B$, $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{2}$, $\alpha_3 = \gamma = 0$, $\alpha_4 = \frac{1}{\sqrt{2}}$ and $\alpha_5 = \frac{1}{\sqrt{2}}$. Thus (5.22) gives $h = 0$. Since $h = \alpha_1 + 3\alpha_2 + 3\alpha_3 + (4n-4)(\alpha_4 + \alpha_5)$, we have $4\sqrt{2}n = 0$ which is a contradiction.

- **Case 3.** Exceptional case $\mathcal{E}$

For $X \in T_{\alpha_5} \subset T_\mathcal{E}$, $G(X) = -\frac{1}{2}(\alpha_5 - \alpha_4)(\alpha_5 + \alpha_4)\phi X$. On $T_\mathcal{E}$ we have $\alpha_1 = \alpha = \sqrt{2}$, $\alpha_4 = \lambda = \frac{1}{\sqrt{2}}$ and $\alpha_5 = \mu = \pm\frac{1}{\sqrt{2}}$. Because $\mu \neq \lambda$, it should be $\mu = -\frac{1}{\sqrt{2}}$. Moreover, since $JT_\mu \subset T_\lambda$ and $\mathfrak{J}T_\mu \subset T_\lambda$, we see that the corresponding multiplicities of the eigenvalues λ and μ satisfy $m(\lambda) \geq m(\mu)$. Since $m(\alpha) = 4$, $m(\gamma) = 3$ and $m(\lambda) + m(\mu) = 4m - 8$ on $\mathcal{E}$, the trace of the shape operator A denoted by h becomes $h = 4\alpha + 3\gamma + m(\lambda)\lambda + m(\mu)\mu = 4\sqrt{2} + \frac{1}{\sqrt{2}}(m(\lambda) - m(\mu))$, which makes a contradiction. In fact, since we obtained $h = 0$ on $T_\gamma \in T\mathcal{E}$, it yields $(m(\lambda) - m(\mu)) = -8 < 0$. Thus, this case does not occur.

This shows that hypersurfaces of $\mathcal{T}_B$, $\mathcal{H}_B$ or $\mathcal{E}$ cannot satisfy the condition (C-2), and therefore in the situation of Theorem 2, the case $X \in \mathcal{Q}$ cannot occur. This completes the proof of Theorem 2.

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